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1987 J. Phys. A: Math. Gen. 20 537

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J. Phys. A: Math. Gen. 20 (1987) 537-549. Printed in the UK

On the new invariance algebras and superalgebras of relativistic wave equations

W I Fushchich and A G Nikitin

Institute of Mathematics, Academy of Sciences of Ukrainian SSR, Repin Street 3, Kiev-4, USSR

Received 31 October 1985

Abstract. We show that any relativistic wave equation for a particle with mass $m > 0$ and arbitrary spin s is invariant under the Lie algebra of the group $GL(2s+1, C)$. The explicit form of basis elements of this algebra is given for any s. The complete sets of the symmetry operators of the Dirac and Maxwell equations are obtained, which belong to the classes of the first- and second-order differential operators with matrix coefficients. Corresponding new conservation laws and constants of motion are found.

1. Introduction

The classical Lie approach is the main mathematical apparatus used for the analysis of the symmetry of partial differential equations (Ames 1965, Ovsjannikov 1978). This approach was that from which it was established that the Poincaré group is the maximal symmetry group of the Dirac equation (Danilov 1968, Ibragimov 1969) and that the maximal symmetry of Maxwell's equations is determined by the conformal group replenished by the Heaviside-Larmor-Rainich transformation. However, in spite of its power and universality, the Lie approach does not make it possible to find all the symmetry operators of the given equation. Actually it gives the possibility or finding only such symmetry operators which are the first-order differential operators.

Using the non-Lie approach (Fushchich 1970, 1971, 1974, 1978), in which the invariance group generators may be differential operators of any order and even integro-differential operators, the new invariance groups of a number of relativistic wave equations have been found. It has been demonstrated that the Dirac equation was invariant under the group $SU(2)\times SU(2)$ (Fushchich 1970, 1971, Fushchich and Nikitin 1977) and that the Kemmer-Duffin-Petiau equation for the vector field was invariant under the group $SU(3)\times SU(3)$ (Nikitin *et al* 1976, Fushchich and Nikitin 1977). The non-Lie approach gave the possibility of finding the additional symmetry of the Dirac and Kemmer-Duffin-Petiau equations describing the particles in an external electromagnetic field (Fushchich and Nikitin 1978, Nikitin 1978). The hidden symmetry of Maxwell's equations has also been found and is described by the eight-parameter transformation group including the subgroup of Heaviside-Larmor-Rainich transformations (Fushchich and Nikitin 1978, 1979a, b, 1983).

In this paper we continue to study the symmetry of the Dirac, Weyl and Maxwell equations and of relativistic wave equations for any spin particles. The main results obtained here may be formulated as follows.

(i) We found that any Poincaré-invariant wave equation for a particle of arbitrary spin s and mass $m=0$ is additionally invariant under the $2(2s+1)(2s+1)$-dimensional Lie algebra which is isomorphic to the Lie algebra of the group $GL(2s+1, C)$. The explicit form of basis elements of this invariance algebra is found for any value of s. Thus the additional symmetry of an arbitrary relativistic wave equation is described whereas previously one studied, as a rule, the symmetry properties of specific equations.

(ii) In our earlier work we restricted ourselves to studying symmetry operators of relativistic wave equations which belong to a finite-dimensional Lie algebra (Fushchich 1983). Here we also consider the symmetry operators belonging to the classes of first- and second-order differential operators with matrix coefficients which, generally speaking, are not the basis elements of any finite-dimensional Lie algebra, but are closely connected with conservation laws. The complete set of symmetry operators of the Dirac equation in the class of first-order differential operators with matrix coefficients (class $\mathfrak{M}_1$) is found. We also obtain the symmetry operators of the Weyl and Maxwell equations which form the basis of the Lie superalgebra.

(iii) The new conservation laws and motion constants, which are connected with hidden symmetry of the Dirac and Maxwell equations, are found.

The results of this paper supplement and in some sense complete, those obtained by us and expanded by a number of other authors (Da Silveira 1980, Stražev 1981, Kotelnikov 1982, Stražev and Shkolnikov 1984) by studying the additional symmetry of Poincaré-invariant wave equations.

2. The additional symmetry of Poincaré-invariant wave equations for arbitrary spin particles

In this section we demonstrate that any relativistic wave equation for a particle of non-zero mass and spin $s=0$ has more extensive symmetry than Poincaré invariance, and describe this additional symmetry exactly.

Let us write an arbitrary linear (differential or integro-differential) equation in the following symbolic form

$$L\psi=0 \tag{2.1}$$

where L is a linear operator defined on a vector space H, $\psi \in H$.

Let Q be an operator defined on H. We say that Q is the symmetry operator of the equation (2.1), if

$$L(Q\psi)=0 \tag{2.2}$$

for any ψ satisfying (2.1).

Definition. Equation (2.1) is Poincaré-invariant and describes a particle of mass m and spin s if it has 10 symmetry operators P_μ, $J_{\mu\nu}$, μ, $\nu=0,1,2,3$, which form the basis of the Lie algebra of the Poincaré group, and any solution ψ satisfies the conditions

$$P_\mu P^\mu \psi = m^2\psi \qquad W_\mu W^\mu \psi = -m^2 s(s+1)\psi \tag{2.3}$$

where W_μ is the Lubansky-Pauli vector

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma. \tag{2.4}$$

Below we consider only such equations (2.1) which satisfy the given definition and so may be interpreted as equations for a relastivistic particle of spin s and mass m. The symmetry operators P_μ, $J_{\mu\nu}$ of such a equation satisfy the commutation relations

$$\begin{aligned} &[P_\mu, P_\nu]=0 \qquad [P_\mu, J_{\nu\sigma}]=\mathrm{i}(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu) \\ &[J_{\mu\nu}, J_{\lambda\sigma}]=\mathrm{i}(g_{\mu\sigma}J_{\nu\lambda}+g_{\nu\lambda}J_{\mu\sigma}-g_{\mu\lambda}J_{\nu\sigma}-g_{\nu\sigma}J_{\mu\lambda}) \end{aligned} \tag{2.5}$$

which characterise the Lie algebra of the Poincaré group P(1,3). The eigenvalues of the corresponding Casimir operators $P_\mu P^\mu$ and $W_\mu W^\mu$ are fixed and given by the relations (2.3). Let us emphasise that we do not make any supposition with regards to the explicit form of the operators P_μ and $J_{\mu\nu}$—they can be as differential operators of first order as non-local (integro-differential) ones.

Theorem 1. Any Poincaré-invariant equation for a particle of mass m and spin s is invariant under the algebra† GL $(2s+1, C)$.

Proof. Let P_μ, $J_{\mu\nu}$ be the symmetry operators of the equation (2.1) satisfying the commutation relations (2.5). Then by the definition (2.3) the following combinations

$$Q^{\pm}_{\mu\nu}=\frac{1}{m^2}[\varepsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma \pm \mathrm{i}(P_\mu W_\nu - P_\nu W_\mu)] \tag{2.6}$$

are also the symmetry operators of these equations.

Using (2.5) and the relations

$$[W_\mu, P_\nu]=0 \qquad [W_\mu, W_\nu]=\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}P^\rho W^\sigma \tag{2.7}$$

can make sure that the operators (2.6) satisfy the conditions

$$[Q^{\pm}_{\mu\nu}, Q^{\pm}_{\lambda\sigma}]=\mathrm{i}(g_{\mu\sigma}Q^{\pm}_{\nu\lambda}+g_{\nu\lambda}Q^{\pm}_{\mu\sigma}-g_{\mu\lambda}Q^{\pm}_{\nu\sigma}-g_{\nu\sigma}Q^{\pm}_{\mu\lambda})m^{-4}(P_\mu P^\mu)^2 \tag{2.8}$$

$$\begin{aligned} &C_1=\tfrac{1}{4}Q^{\pm}_{\mu\nu}Q^{\pm\mu\nu}=-m^{-4}W_\lambda W^\lambda P_\sigma P^\sigma \\ &C_2=\tfrac{1}{4}\varepsilon_{\mu\nu\rho\sigma}Q^{\pm\mu\nu}Q^{\pm\rho\sigma}=\mp \mathrm{i}m^{-4}W_\mu W^\mu P_\sigma P^\sigma. \end{aligned} \tag{2.9}$$

It follows from (2.3) and (2.8) that on the set of the equation (2.1) solutions the operators (2.6) satisfy the commutation relations

$$[Q^{\pm}_{\mu\nu}, Q^{\pm}_{\lambda\sigma}]\psi=\mathrm{i}(g_{\mu\sigma}Q^{\pm}_{\nu\lambda}+g_{\nu\lambda}Q^{\pm}_{\mu\sigma}-g_{\mu\lambda}Q^{\pm}_{\nu\sigma}-g_{\nu\sigma}Q^{\pm}_{\mu\lambda})\psi \tag{2.10}$$

which characterise the Lie algebra of the group SL(2, C). From (2.3) and (2.9) one obtains the eigenvalues of corresponding Casimir operators

$$C_1\psi=\tfrac{1}{2}(l_0^2+l_1^2-1)\psi \qquad C_2\psi=\mathrm{i}l_0 l_1\psi \tag{2.11}$$

where $l_0=s$, $l_1=\pm(s+1)$.

So we have demonstrated that any Poincaré-invariant equation for a particle of non-zero mass and spin $s\neq 0$ is additionally invariant under the algebra SL(2, C), the basis elements of which belong to the enveloping algebra of the algebra P(1, 3) and are given exactly by the relations (2.6). According to (2.11) the operators (2.6) realise the representation $D(l_0, l_1)=D(s, \pm(s+1))$ of the algebra GL(2, C). Now we see that

† We use the same notation for the groups and for the corresponding Lie algebras.

this invariance algebra may be extended to $2(2s+1)$-dimensional Lie algebra isomorphic to the algebra GL$(2s+1, C)$. Exactly the basis elements of the algebra GL$(2s+1, C)$ have the following form on the set of the equation (2.1) solutions:

$$\begin{aligned}
&\lambda_{n+k\,n} = a_{kn}(Q^+_{23} - Q^+_{02})^k P^s_n \\
&\lambda_{n\,n+k} = a_{kn} P^s_n (Q^+_{23} + Q^+_{02}) \\
&\tilde{\lambda}_{mn} = Q_1 \lambda_{mn}
\end{aligned} \tag{2.12}$$

where

$$P^s_n = \prod_{n' \neq n} \frac{Q_{12} - s - 1 + n'}{n' - n} \qquad Q_1 = \frac{\varepsilon_{abc}}{2s(s+1)} Q^+_{0a} Q^+_{bc}$$

$$m, n = 1, 2, \ldots 2s+1 \qquad k = 0, 1, \ldots, 2s - n$$

and a_{kn} are the coefficients determined by the recurrent relations

$$a_{0n} = 1 \qquad a_{1n} = [n(2s+1-n)]^{-1/2}$$

$$a_{\lambda n} = a_{\lambda-1\,n} a_{\lambda-1\,n+\lambda-1} \qquad \lambda = 2, 3, \ldots, 2s - n.$$

Actually the polynomials of the symmetry operators $Q^+_{\mu\nu}$ given by the relations (2.12) manifestly are the symmetry operators of equation (2.1). Operators (2.11) form the basis of the algebra GL$(2s+1, C)$ inasmuch as they satisfy the following commutation relations

$$\begin{aligned}
&[\lambda_{ab}, \lambda_{cd}] = -[\tilde{\lambda}_{ab}, \lambda_{cd}] = \delta_{bc}\lambda_{ad} - \delta_{ad}\lambda_{bc} \\
&[\lambda_{ab}, \tilde{\lambda}_{cd}] = \delta_{bc}\tilde{\lambda}_{ad} - \delta_{ad}\tilde{\lambda}_{bc} \qquad a, b, c, d = 1, 2, \ldots, 2s+1
\end{aligned} \tag{2.13}$$

which characterise the algebra GL$(2s+1, C)$. The relations (2.13) are correct on the set of the equation (2.1) solutions. The validity of these solutions can be verified by direct calculation using the equivalent matrix representation for the basis elements of the algebra SL$(2, C)$ (which is evaluated according to (2.11))

$$Q^+_{ab} = \varepsilon_{abc} S_c \qquad Q^+_{0a} = -\mathrm{i} S_a.$$

Here S_a are the matrices which realise the representation $D(s)$ of the SO(3) algebra in the Gelfand-Zetlin basis (Gelfand and Zetlin 1950). Thus the theorem is proved.

So if equation (2.1) is Poincaré invariant and describes a particle of spin s and mass $m > 0$, it is invariant also under the algebra GL$(2s+1, C)$, the basis elements of which belong to the enveloping algebra of the algebra P(1, 3). The operators (2.12) together with the Poincaré generators P_μ and $J_{\mu\nu}$ form the basis of the $10 + 2(2s+1)$-dimensional Lie algebra isomorphic to the algebra P$(1,3) \oplus$GL$(2s+1, C)$. The last statement can easily be verified by moving to the new basis $P_\mu \to P_\mu$, $J_{\mu\nu} \to J_{\mu\nu} - Q_{\mu\nu}$, $\lambda_{mn} \to \lambda_{mn}$, $\tilde{\lambda}_{mn} \to \tilde{\lambda}_{mn}$, where

$$Q_{12} = \sum_n (s - n + 1)\lambda_{nn} \qquad Q_{03} = \sum_n (s - n + 1)\tilde{\lambda}_{nn}$$

$$Q_{23} = \sum_n \frac{1}{2a_{1n}}(\lambda_{n\,n+1} + \lambda_{n+1\,n}) \qquad Q_{31} = -\mathrm{i}[Q_{12}, Q_{23}]$$

$$Q_{02} = \mathrm{i}[Q_{23}, Q_{03}] \qquad Q_{01} = -\mathrm{i}[Q_{31}, Q_{03}].$$

The theorem proved has a constructive character insofar as it gives the explicit form of the basis elements of additional invariance algebra via the Poincaré generators. Starting, for example, from the Poincaré generators for the Dirac equation

$$P_\mu = p_\mu = \mathrm{i}\frac{\partial}{\partial x^\mu} \qquad J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu + \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu] \tag{2.14}$$

where γ_μ are the Dirac matrices, one obtains by the formula (2.6) the additional symmetry operators of this equation found earlier by Fushchich and Nikitin (1977). In an analogous way to formulae (2.6) and (2.12), the additional invariance algebras of the Kemmer-Duffin-Petiau and Proca equations may be obtained (see Fushchich and Nikitin 1977, 1983, Fushchich and Vladimirov 1981, 1983) and even the invariance algebra of infinite-component wave equations (Fushchich and Onufrijchuck 1977) may be found.

Let us note that relativistic wave equations for a particle of spin $s > 0$ also possess such additional invariance algebras which belong to the class of integro-differential operators (Fushchich 1970, 1974, 1978, Fushchich and Nikitin 1982, 1983, Nikitin *et al* 1976, Nikitin 1978) and, generally speaking, are not numbered among the enveloping algebras of the algebra P(1, 3).

3. Symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$

Here we consider in detail the symmetry properties of the Dirac equation

$$L\psi \equiv (\gamma^\mu p_\mu - m)\psi = 0. \tag{3.1}$$

It is well known that the symmetry of equation (3.1) which can be found in the classical Lie approach is exhausted by invariance under the algebra P(1, 3), the basis elements of which are given in (2.14), and under a corresponding group of transformations, i.e. the Poincaré group.

Theorem 1 gives the possibility of extending the set of symmetry operators of the Dirac equation. Actually, using formulae (2.6), (2.14) and (3.1) one obtains the additional symmetry operators (Fushchich and Nikitin 1977, 1983)

$$Q^{\pm}_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu] + \frac{\mathrm{i}}{2m}(\gamma_\mu p_\nu - \gamma_\nu p_\mu)(1 \pm \mathrm{i}\gamma_4). \tag{3.2}$$

The operators (3.2) are the first-order differential operators with matrix coefficients (i.e. belong to the class $\mathfrak{M}_1$) and so they cannot be found in the frames of classical Lie approach. But these operators (with fixed sign $\pm$) form the basis of 16-dimensional Lie algebra together with the Poincare generators (2.14). It follows from the above that the Dirac equation is invariant under the 16-parameter group including the Lorentz transformations (generated by P_μ, $J_{\mu\nu}$) and the transformations which are generated by the operators (3.2). Specifically these transformations have the form

$$\psi \to \psi' = \exp(2\mathrm{i}\theta Q)\psi = (\cos\theta - \gamma_1\gamma_2 \sin\theta)\psi$$
$$\times \frac{\mathrm{i}}{m}(1 \mp \mathrm{i}\gamma_4)\sin\theta\left(\gamma_1\frac{\partial\psi}{\partial x_2} - \gamma_2\frac{\partial\psi}{\partial x_1}\right)$$

if $Q = Q^{\mp}_{12}$ etc (Fushchich and Nikitin 1977).

It may be interesting to know whether the operators (2.14) and (3.3) exhaust all symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$. It turns out that this is not so.

Here we find the complete set of symmetry operators $Q \in \mathfrak{M}_1$ for equation (3.1) which, however, do not form the basis of Lie algebra.

Theorem 2. The Dirac equation has 26 linearly independent symmetry operators $Q \in \mathfrak{M}_1$. These operators include the Poincaré generators (2.14), identity operator and fifteen operators given below

$$\begin{aligned} \eta_\mu &= \tfrac{1}{4}\mathrm{i}\gamma_4(p_\mu - m\gamma_\mu) \\ \omega_{\mu\nu} &= mS_{\mu\nu} + \tfrac{1}{2}\mathrm{i}(\gamma_\mu p_\nu - \gamma_\nu p_\mu) \\ A_\mu &= \omega_{\mu\nu}x^\nu + x^\nu\omega_{\mu\nu} - \mathrm{i}\gamma_\mu \\ B &= \mathrm{i}\gamma_4(D - m\gamma_\mu x^\mu) \end{aligned} \tag{3.3}$$

where

$$D = x^\mu p_\mu + \tfrac{3}{2}\mathrm{i} \qquad S_{\mu\nu} = \tfrac{1}{4}\mathrm{i}[\gamma_\mu, \gamma_\nu] \qquad \mu, \nu = 0, 1, 2, 3. \tag{3.3'}$$

Proof. To find all linearly independent symmetry operators of the Dirac equation in the class $\mathfrak{M}_1$ it is necessary to obtain the general solution of the following operator equations

$$[L, Q] = f_Q L \tag{3.4}$$

where $L = \gamma^\mu p_\mu - m$, Q and f_Q are unknown operators belonging to $\mathfrak{M}_1$:

$$Q = \tilde{A}^\mu p_\mu + \tilde{B} \qquad f_Q = \tilde{C}^\mu p_\mu + \tilde{D}$$

$\tilde{A}_\mu$, $\tilde{B}_\mu$, $\tilde{C}_\mu$ and $\tilde{D}$ are 4×4 matrices depending on $x = (x_0, \boldsymbol{x})$.

Relations (3.4) mean that the operators on the RHS and LHS give the same result acting on arbitrary solutions of the Dirac equation. On the set of these solutions operator p_0 can be expressed via the operators p_a with matrix coefficients: $p_0\psi = H\psi \equiv (\gamma_0 m + \gamma_0\gamma_a p_a)\psi$. In other words it is sufficient to restrict ourself by considering symmetry operators of a form such that

$$Q = \boldsymbol{B}\cdot\boldsymbol{p} + G \tag{3.5}$$

where $\boldsymbol{B}$ and G are 4×4 matrices depending on x. For the operators (3.5) the invariance condition (3.4) reduces to the following form:

$$[p_0 - H, Q] = f_Q(p_0 - H) \tag{3.6}$$

where $f_Q \equiv 0$ insofar as the commutator on the LHS cannot depend on p_0.

An unknown operator (3.5) can be expanded via a complete set of the Dirac matrices

$$\begin{aligned} \boldsymbol{B} &= I\boldsymbol{d}^0 + \mathrm{i}\gamma_4\boldsymbol{d}^1 + \gamma_\nu\boldsymbol{n}^\nu + S_{\mu\nu}\boldsymbol{m}^{\mu\nu} + \gamma_4\gamma_\nu\boldsymbol{b}^\nu \\ G &= I\alpha^0 + \mathrm{i}\gamma_4 a^1 + \gamma_\nu c^\nu + S_{\mu\nu}f^{\mu\nu} + \gamma_4\gamma_\nu g^\nu \end{aligned} \tag{3.7}$$

where $\boldsymbol{d}^0$, $\boldsymbol{d}^1$, $\boldsymbol{n}^\nu$, $\boldsymbol{m}^{\mu\nu}$, $\boldsymbol{b}^\nu$, a^0, a^1, c^ν, $f^{\mu\nu}$, g^ν are unknown functions on x.

Substituting (3.5) and (3.7) into (3.6) and equating coefficients by the linearly independent matrices and differential operators one comes to the following system of partial differential equations:

$$n^0 = b^0 = 0 \qquad n_b^a = \mathrm{i}\varepsilon_{abc} d_c^2 \qquad b_b^a = \mathrm{i}\varepsilon_{abc} d_c^3$$
$$m_b^{0a} = \mathrm{i}\delta_{ab} A^0 \qquad m_c^{ab} = \varepsilon_{abc} A^1 \qquad a, b, c = 1, 2, 3 \tag{3.8}$$

$$\frac{\partial d_a^\mu}{\partial x_b} = -\frac{\partial d_b^\mu}{\partial x_a} \qquad \frac{\partial d_a^\mu}{\partial x_a} = \frac{\partial d_b^\mu}{\partial x_b} \qquad a \neq b$$
$$m \operatorname{div} \boldsymbol{d}^0 = 0 \qquad m \operatorname{div} \boldsymbol{d}^1 = 2\mathrm{i} m a^1$$
$$\dot{\boldsymbol{d}}^2 = -\tfrac{1}{2} \operatorname{rot} \boldsymbol{d}^3 \qquad \dot{\boldsymbol{d}}^3 = \tfrac{1}{2} \operatorname{rot} \boldsymbol{d}^2$$
$$\dot{\boldsymbol{d}}^i = -\operatorname{grad} A^i \qquad \operatorname{div} \boldsymbol{d}^i = -3\dot{A}^i \qquad i = 0, 1$$
$$c_0 = -\tfrac{1}{3} \operatorname{div} \boldsymbol{d}^3 + m A^0 \qquad c^a = -\tfrac{1}{2}(\operatorname{rot} \boldsymbol{d}^2)_a$$
$$g^0 = \tfrac{1}{3} \operatorname{div} \boldsymbol{d}^2 \qquad g^a = -\tfrac{1}{2}(\operatorname{rot} \boldsymbol{d}^3)_a - \mathrm{i} m d_a^1 \tag{3.9}$$
$$\dot{a}^0 = -\tfrac{1}{2}\mathrm{i} \operatorname{div} \boldsymbol{d}^0 \qquad \operatorname{grad} a^0 = -\tfrac{3}{2}\mathrm{i}\ddot{\boldsymbol{d}}^0$$
$$a^1 = -\tfrac{1}{2}\mathrm{i} \operatorname{div} \dot{\boldsymbol{d}}^1 + \tfrac{1}{3} m \operatorname{div} \boldsymbol{d}^2 \qquad \operatorname{grad} a^1 = -m\dot{\boldsymbol{d}}^2 - \tfrac{3}{2}\mathrm{i}\ddot{\boldsymbol{d}}^1$$
$$f^{0a} = \tfrac{1}{2}\dot{d}_a^0 - \tfrac{1}{4}\mathrm{i}(\operatorname{rot} \boldsymbol{d}^1)_a$$
$$f^{ab} = \varepsilon_{abc}[\tfrac{1}{2}\mathrm{i}\dot{d}_c^1 + \tfrac{1}{4}(\operatorname{rot} \boldsymbol{d}^0)_c + m d_c^2]$$

where the dot denotes the derivative on x_0 and there is no sum by the repeated indices. The symbol d^μ denotes a vector with components $(d_1^\mu, d_2^\mu, d_3^\mu)$ (analogous notation is used for other vector quantities).

The first line in (3.9) gives the equations in the Killing form. Using this circumstance it is not difficult to obtain the general solution of the system (3.9) for $m \neq 0$:

$$\boldsymbol{d}^0 = \boldsymbol{x} \times \boldsymbol{\eta} + \boldsymbol{\rho} x_0 + \boldsymbol{\nu} \qquad \boldsymbol{d}^1 = \boldsymbol{\xi} + \lambda \boldsymbol{x}$$
$$\boldsymbol{d}^2 = \boldsymbol{x} \times \boldsymbol{\varepsilon} + \boldsymbol{\zeta} \qquad \boldsymbol{d}^3 = \boldsymbol{\varepsilon} x_0 + \mu \boldsymbol{x} + \boldsymbol{\sigma}$$
$$g^0 = 0 \qquad g^a = -\mathrm{i} m(\xi_a + \lambda x_a) \qquad f^{0a} = \tfrac{1}{2}\rho_a$$
$$f^{ab} = \tfrac{1}{2}\varepsilon_{abc}(2m\zeta_c - \eta_c) + m(x_a \varepsilon_b - x_b \varepsilon_a) \tag{3.10}$$
$$c^0 = -\mu - m(\boldsymbol{\rho} \cdot \boldsymbol{x} + \varkappa) \qquad c^a = \varepsilon^a \qquad A^0 = -\boldsymbol{\rho} \cdot \boldsymbol{x} - \varkappa$$
$$A^1 = -\lambda x_0 + \omega \qquad a^0 = \Omega \qquad a^1 = -\tfrac{3}{2}\mathrm{i}\lambda.$$

Here the Greek letters denote arbitrary constants.

So the general solution of the system (3.9) depends on 26 arbitrary numerical parameters. Substituting (3.7), (3.8) and (3.10) into (3.5) and using equation (3.1), one obtains a general expression for the symmetry operator $Q \in \mathfrak{M}_1$ for the Dirac equation as a linear combination of the Poincaré group generators (2.14), identity operator and the operators (3.3). The theorem is proved.

So we have obtained the complete set of the symmetry operators $Q \in \mathfrak{M}_1$ for the Dirac equation with $m \neq 0$. Besides the Poincaré group generators (2.14) this set includes four operators which coincide on the set of the equation (3.1) solutions with Lubansky-Pauli vector (2.4), six operators $\omega_{\mu\nu} = \tfrac{1}{2}(Q_{\mu\nu}^+ + Q_{\mu\nu}^-)$, trivial identity operator and five symmetry operators B and A_μ, $\mu = 0, 1, 2, 3$, which belong to the enveloping algebra generated by the Poincaré generators.

The operators (3.3) satisfy the following commutation relations

$$[B, P_\mu] = -2\mathrm{i}\eta_\mu \qquad [B, \eta_\mu] = -\tfrac{1}{2}\mathrm{i}(P_\mu + mA_\mu)$$

$$[A_\mu, P_\nu] = \frac{1}{m}[\eta_\mu, \eta_\nu] = -2\mathrm{i}\omega_{\mu\nu}.$$

However these operators do not form the basis of the Lie algebra inasmuch as the commutators $[\omega_{\mu\nu}, \omega_{\lambda\sigma}]$ do not belong to the class $\mathfrak{M}_1$.

One of the most interesting consequences of the symmetry described in theorem 2 is the existence of new conservation laws for the Dirac equation. Corresponding new conserved currents have the form

$$\eta_{\nu\mu} = \tfrac{1}{4}\left(\bar{\psi}\gamma_4\gamma_\nu\frac{\partial\psi}{\partial x^\mu} - \frac{\partial\psi}{\partial x^\mu}\gamma_\nu\gamma_4\psi\right) + m\bar{\psi}\gamma_4 S_{\mu\nu}\psi$$

$$\omega_{\mu\rho\nu} = \tfrac{1}{4}\mathrm{i}\left(\frac{\partial\bar{\psi}}{\partial x^0}S_{\nu\mu}\psi + \bar{\psi}S_{\nu\rho}\frac{\partial\psi}{\partial x^\mu} - \bar{\psi}S_{\nu\mu}\frac{\partial\psi}{\partial x^\rho} - \frac{\partial\bar{\psi}}{\partial x^\mu}S_{\nu\rho}\psi\right) + \tfrac{1}{2}m\bar{\psi}[S_{\mu\nu}, \gamma_\lambda]_+\psi \tag{3.11}$$

$$B_\nu = 2x^\mu\eta_{\mu\nu} \qquad A_{\mu\nu} = 2x^\lambda\omega_{\mu\lambda\nu}.$$

The tensors $\eta_{\mu\nu}$, $\omega_{\mu\rho\nu}$, $A_{\mu\nu}$ and the vector B_ν correspond to the symmetry operators η_μ, $\omega_{\mu\rho}$, A_μ and B. All quantities (3.11) satisfy the continuity equations

$$p^\nu\eta_{\mu\nu} = 0 \qquad p^\nu\omega_{\mu\rho\nu} = 0 \qquad p^\nu A_{\mu\nu} = 0 \qquad p^\nu B_\nu = 0$$

and so generate conservation laws.

4. Additional symmetry of the Weyl and massless Dirac equations

Here we study the symmetry of the Weyl equation

$$\sigma^\mu p_\mu\varphi = 0 \tag{4.1}$$

where φ is the two-component spinor and σ^μ the Pauli matrices. Putting

$$\psi = \begin{pmatrix} \varphi + \varphi^* \\ \mathrm{i}(\varphi^* - \varphi) \end{pmatrix} \tag{4.2}$$

one may rewrite this equation in the Dirac form

$$\gamma^\mu p_\mu\psi = 0 \tag{4.3}$$

where γ^μ are the Dirac matrices in the Majorana representation. So we consider the symmetry properties of equation (4.3) in order to obtain the results which are valued as for the Weyl equation as for the massless Dirac one.

Theorem 3. The massless Dirac equation has 46 symmetry operators $Q \in \mathfrak{M}_1$. These operators are

$$P_\mu, \quad J_{\mu\nu}, \quad K_\mu, \quad D, \quad F \qquad FP_\mu, \quad FJ_{\mu\nu}, \quad FK_\mu, \quad FD, \quad I \tag{4.4a}$$

$$\hat{A}_\mu = \hat{\omega}_{\mu\nu}x^\mu + x^\nu\hat{\omega}_{\mu\nu} - \gamma_\mu \qquad \hat{\omega}_{\mu\nu} = \gamma_\mu p_\nu - \gamma_\nu p_\mu \qquad F\hat{A}_\mu \tag{4.4b}$$

where $K_\mu = 2x_\mu D - p_\mu x_\nu x^\nu + 2S_{\mu\nu}x^\nu$, $F = \mathrm{i}\gamma_4$; P_μ, $J_{\mu\nu}$ and D are given in (2.14) and (3.5′).

Proof. This can be carried out in full analogy with the proof of theorem 2. The general solution of the system (3.8) for the case $m=0$ has the form

$$
\begin{aligned}
d^\alpha &= xx\cdot\boldsymbol{\mu}^\alpha+\tfrac{1}{2}\mu^\alpha(x_0^2-x^2)+x\times\boldsymbol{\eta}^\alpha+\nu^\alpha x+\rho^\alpha x_0 x+\lambda^\alpha+\omega^\alpha x_0 \qquad \alpha=0,1,\\
d^2 &= x\times\boldsymbol{\varepsilon}+\xi x-\boldsymbol{\zeta}x_0+\boldsymbol{\varphi}\\
d^3 &= x\times\boldsymbol{\zeta}+\sigma x+\boldsymbol{\varepsilon}x_0+\boldsymbol{\varkappa}\\
A^\alpha &= -[x\cdot\boldsymbol{\mu}^\alpha x_0+\tfrac{1}{2}\rho^\alpha(x_0^2+x^2)+\nu^\alpha x_0+\boldsymbol{\omega}^\alpha x+\chi^\alpha]\\
a^\alpha &= -\tfrac{3}{2}\mathrm{i}(x\cdot\boldsymbol{\mu}^\alpha+\rho^\alpha x_0+\delta^\alpha)\\
c^0 &= \sigma \qquad c^a=-\varepsilon_a \qquad g^0=\xi \qquad g^a=-\zeta_a\\
f^{0a} &= \tfrac{1}{2}(-\eta_a^1+\rho^0 x_a+\mu_a^0 x_0+\omega_a^0)\\
f^{ab} &= -\tfrac{1}{2}\varepsilon_{abc}(\mu_c^1 x_0+\rho_1^1 x_c+\omega_c^1+\eta_c^0)
\end{aligned}
\tag{4.5}
$$

and includes 46 independent parameters denoted by the Greek letters. Substituting (3.5) and (4.5) into (3.7) and using equation (4.3) one obtains a general expression for the symmetry operator of the massless Dirac equation in the form of a linear combination of the operators (4.4). Thus the theorem is proved.

Among the operators (4.4) there are exactly fourteen symmetry operators, which do not belong to the enveloping algebra generated by the conformal group generators P_μ, $J_{\mu\nu}$, K_μ, D and by the operator $F=\mathrm{i}\gamma_4$. These essentially new symmetry operators are given in (4.4b).

Operators (4.4) transform the real wavefunction ψ (4.2) into real wavefunction $\psi'=Q\psi$ and so they are also the symmetry operators for the Weyl equation (4.1). Incidentally the linear transformations of ψ (4.2) generate linear and antilinear transformations of a two-component Weyl spinor.

The operators (4.4) do not form a basis of the Lie algebra. However, one may consider different subsets of the operators (4.4) which have the structure of the Lie algebra or superalgebra. Thus the operators (4.4a) form the basis of 32-dimensional Lie algebra including the Lie algebra of the conformal group. The operators $J_{\mu\nu}$, $\hat\omega_{\mu\nu}$, F and $\lambda_\mu=FP_\mu$ satisfy the following commutation and anticommutation relations:

$$
\begin{aligned}
&[\hat\omega_{\mu\nu},\hat\omega_{\lambda\sigma}]_+=-2\mathrm{i}[J_{\mu\nu},p_\lambda p_\sigma]=2(g_{\mu\lambda}p_\nu p_\sigma+g_{\nu\sigma}p_\mu p_\lambda-g_{\mu\sigma}p_\nu p_\lambda-g_{\nu\lambda}p_\mu p_\sigma)\\
&[J_{\mu\nu},\hat\omega_{\lambda\sigma}]=\mathrm{i}(g_{\mu\sigma}\hat\omega_{\nu\lambda}+g_{\nu\lambda}\hat\omega_{\mu\sigma}-g_{\mu\lambda}\hat\omega_{\nu\sigma}-g_{\nu\sigma}\hat\omega_{\mu\lambda})\\
&[\hat\omega_{\mu\nu},\lambda_\rho]_+=[\omega_{\mu\nu},F]_+=0 \qquad F^2=I\\
&[\lambda_\mu,\lambda_\nu]_+=2p_\mu p_\nu \qquad [\lambda_\mu,F]_+=2P_\mu
\end{aligned}
\tag{4.6}
$$

where the symbol $[A,B]_+$ denotes the anticommutator $[A,B]_+=AB+BA$.

It follows from (2.5) and (4.6) that the set of symmetry operators $\{P_\mu, J_{\mu\nu}, p_\mu p_\nu, I;\ F, \lambda_\mu, \hat\omega_{\mu\nu}\}$ form the basis of the Lie superalgebra (which includes ten symmetry oerators $p_\mu p_\nu$ not belonging to the class $\mathfrak{M}_1$).

5. The symmetry and supersymmetry of Maxwell's equations

We shall write Maxwell's equations for the electromagnetic field in vacuum in the following form (Fushchich and Nikitin 1983):

$$
\begin{aligned}
L_1\psi &\equiv (i\,\partial/\partial x_0+\sigma_2 \boldsymbol{S}\cdot\boldsymbol{p})\psi=0\\
L_2^a\psi &\equiv (p_a-\boldsymbol{S}\cdot\boldsymbol{p}p_a)\psi=0.
\end{aligned}
\tag{5.1}
$$

Here

$$\sigma_2 = \mathrm{i}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad S_a = \begin{pmatrix} s_a & 0 \\ 0 & s_a \end{pmatrix} \tag{5.2}$$

where 1 and 0 are unit and zero 3×3 matrices, s_a are the generators of irreducible representation D(1) of the group SO(3) with the matrix elements $(s_a)_{bc} = \mathrm{i}\varepsilon_{abc}$. The symbol ψ denotes the six-component function, ψ = column $(E_1, E_2, E_3, H_1, H_2, H_3)$, where E_a and H_a are the components of the vectors of electric and magnetic fields strengths.

It is well known that the Maxwell equations are invariant under the conformal group C(1, 3) and under the group H of Heaviside–Larmor–Rainich transformations. Moreover it was found (Fushchich and Nikitin 1979a, b, 1982, 1983) that these equations also have the additional hidden symmetry in the class of integro-differential operators which is determined by the algebra GL(2, C). It was demonstrated also that GL(2, C) is the most extensive invariance algebra of the Maxwell equations if one supposes the symmetry operators do not depend on x.

Here we study the symmetry of the Maxwell equations in quite another aspect. The requirement that the symmetry operators belong to a finite-dimensional Lie algebra is very important if one is interested in studying the symmetry groups of the equations considered. However for many applications (e.g. for constructing conservation laws) this requirement is not essential. So we do not require that the symmetry operators of Maxwell's equations should belong to a finite-dimensional Lie algebra but restrict the class of operators considered by the second-order differential operators with constant matrix coefficients. In other words we consider the symmetry operators of a form such that

$$Q = d_{ab}p_a p_b + c_b p_b + g \qquad a, b = 1, 2, 3 \tag{5.3}$$

where d_{ab}, c_6 and g are 6×6 numerical matrices. The operators (5.3) do not depend on p_0 inasmuch as one may always express $p_0\psi$ via $\sigma_2 S\cdot p\psi$ according to equation (5.1). Let us denote the class of the operators (5.3) by the symbol $\mathfrak{M}_2$.

We shall see that the Maxwell equations have non-trivial symmetry operators in the class $\mathfrak{M}_2$ which do not belong to the enveloping algebra of the conformal group generators. On the other hand the analysis of more extensive classes of the Maxwell equation symmetry operators is very complicated and cannot be carried out within the framework of the present paper.

The invariance condition for equation (5.1) under the operators (5.3) may be written in the following general form (Fushchich and Nikitin 1983)

$$\begin{aligned} [L_1, Q] &= \alpha_Q^1 L_1 + \beta_Q^{1a} L_2^a \\ [L_2^a, Q] &= \alpha_Q^{2a} L_1 + \beta_Q^{2a,d} L_2^d \end{aligned} \tag{5.4}$$

where in our case $\alpha_Q^1 = \alpha_Q^{2a} \equiv 0$ since the commutators on the LHS cannot depend on p_0, and

$$\beta_Q^{1a} = g_{bc}^a p_b p_c + f_b^a p_b + h^a \qquad \beta_Q^{2a,d} = g_{bc}^{a,d} p_b p_c + f_c^{a,d} p_c + h^{a,d} \tag{5.5}$$

where g_{bc}^k, f_b^k, h^k are numerical matrices, $k = a$ or $k = a, d$.

Any of the matrices in (5.3) and (5.5) can be represented as a linear combination of the matrices D_c^ν and G_{cd}^ν, where

$$D_c^\nu = \sigma_\nu S_c \qquad G_{cd}^\nu = \sigma_\nu(\delta_{cd} - S_c S_d - S_d S_c).$$

Here σ_ν are the 6×6 Pauli matrices commuting with S_α of (5.2). Calculating the commutators in (5.4) and equating the coefficients by the linearly independent matrices and differential operators one may prove the following statement.

Theorem 4. The Maxwell equations (5.1) have ten linearly independent symmetry operators $Q \in \mathfrak{M}_2$ which do not belong to the enveloping algebra of the Lie algebra of the group $C(1,3)\otimes H$. These operators have the form

$$Q_{ab} = \sigma_1 q_{ab} \qquad \tilde{Q}_{ab} = \sigma_3 q_{ab} \tag{5.6}$$

where

$$q_{ab} = [(\boldsymbol{S}\times\boldsymbol{p})_a, (\boldsymbol{S}\times\boldsymbol{p})_b] - p^2\,\delta_{ab} \qquad p^2 = p_1^2 + p_2^2 + p_3^2.$$

Proof. The proof can be carried out in full analogy with the proofs of theorems 2 and 3 and so can be omitted. We note only that equations (5.3)-(5.5) are satisfied by the 46 linearly independent operators given below:

$$\begin{matrix} \sigma_0 & \mathrm{i}\sigma_0 p_a & \sigma_0 p_a p_b & \sigma_0 \boldsymbol{S}\cdot\boldsymbol{p} & \mathrm{i}p_a \boldsymbol{S}\cdot\boldsymbol{p} & \mathrm{i}\sigma_2 \\ \sigma^2 p_a & \mathrm{i}\sigma_2 p_a p_b & \sigma_2 \boldsymbol{S}\cdot\boldsymbol{p} & \mathrm{i}\sigma_2 p_a \boldsymbol{S}\cdot\boldsymbol{p} & Q_{ab} & \tilde{Q}_{ab} \end{matrix} \tag{5.7}$$

where Q_{ab} and $\tilde{Q}_{ab}$ are given in (5.6). All operators of (5.7) with the exception of Q_{ab} and $\tilde{Q}_{ab}$ can be expressed via P_a, $\boldsymbol{S}\cdot\boldsymbol{p} = \frac{1}{2}\varepsilon_{abc}J_{ab}P_c$ and σ_2, where J_{ab} and P_a are the Poincaré generators given by the formulae (2.14) with $\frac{1}{4}\mathrm{i}[\gamma_a, \gamma_b] \to \varepsilon_{abc}S_c$, σ_2 is the matrix of (5.2) (which is the generator of the Heaviside-Larmor-Rainich transformations).

Note 1. From twenty operators (5.6) exactly ten are linearly independent in so far as

$$(Q_{11} + Q_{22} + Q_{33})\psi = (\tilde{Q}_{11} + \tilde{Q}_{22} + \tilde{Q}_{33})\psi = 0$$

where ψ is an arbitrary solution of equations (5.1).

Note 2. The operators α_Q^1, β_Q^{1a} and α_Q^{2a} from (5.4) which correspond to the symmetry operators (5.6) are zero matrices. For $\beta_{Q_{ab}}^{2c,d}$ and $\beta_{\tilde{Q}_{ab}}^{2c,d}$ one obtains by direct calculation

$$\beta_{Q_{bc}}^{2a,d} = \mathrm{i}\sigma_2 \beta_{\tilde{Q}_{bc}}^{2a,d} = -\sigma_1\,\delta_{ad}[(\boldsymbol{S}\times\boldsymbol{p})_a, (\boldsymbol{S}\times\boldsymbol{p})_b]_+.$$

So we have determined the complete set of the Maxwell equation symmetry operators in the class $\mathfrak{M}_2$. Using the notation given in (5.2) and below formula (5.2), it is not difficult to represent the transformations $\psi \to Q_{ab}\psi$ and $\psi \to \tilde{Q}_{ab}\psi$ generated by operators (5.6), in the terms of the electromagnetic field strengths

$$E_c \to q_{ab}^{cd} H_d \qquad H_c \to q_{ab}^{cd} E_d \tag{5.8}$$

$$E_c \to q_{ab}^{cd} E_d \qquad H_c \to -g_{ab}^{cd} H_d \tag{5.9}$$

where

$$g_{ab}^{cd} = p_a p_b\,\delta_{cd} - p_a p_c\,\delta_{bd} - p_b p_c\,\delta_{ad} + p^2(\delta_{ac}\,\delta_{bd} + \delta_{bc}\,\delta_{ad} - \delta_{ab}\,\delta_{cd}).$$

The invariance of Maxwell's equations under transformations (5.8) and (5.9) can be easily verified by direct calculation.

The operators (5.6) do not form a basis of a Lie algebra. However, one may consider subsets of the operators (5.6) which can be extended to the Lie superalgebras. One of these subsets includes the following operators:

$$Q^1=\tfrac{1}{2}\varepsilon_{abc}c_aQ_{bc} \qquad Q^2=\tfrac{1}{2}\varepsilon_{abc}c_a\tilde{Q}_{bc}$$
$$Q^3=\boldsymbol{S}\cdot\boldsymbol{p} \qquad Q^4=\tfrac{1}{2}c_ac_bp_ap_b \qquad Q^5=p^2 \tag{5.10}$$

where c_a are arbitrary numbers satisfying the condition $c_ac_a=1$. The operators (5.10) satisfy the relations

$$[Q^a,Q^b]_+=2\,\delta_{ab}(Q^a)^2 \qquad (Q^1)^2=(Q^2)^2=Q^6\equiv(Q^4-Q^5)^2$$
$$(Q^3)^2\psi=Q^5\psi \qquad [Q^a,Q^4]=[Q^a,Q^5]=[Q^4,Q^5]=0$$

and so form the basis of the Lie superalgebra together with the operator Q^6. This superalgebra can be extended by adding the operators $Q^{6+a}=\mathrm{i}\sigma^2\boldsymbol{S}\cdot\boldsymbol{p}Q^a$, $Q^{9+a}=\mathrm{i}\sigma^2\boldsymbol{S}\cdot\boldsymbol{p}(Q^a)^2$ and $Q^{12+a}=p^2(Q^a)^2$, $a=1,2,3$, which satisfy the relations

$$[Q^{6+a},Q^{6+b}]_+=2\delta_{ab}Q^{12+b} \qquad [Q^{6+a},Q^b]_+=2\delta_{ab}Q^{6+b}$$
$$[Q^{9+a},Q^B]=[Q^{12+a},Q^B]=0 \qquad B=1,2,\ldots,15.$$

In conclusion let us give the explicit form of the motion constants of the electromagnetic field which correspond to the symmetry operators (5.6). Due to the Maxwell equations the following bilinear combinations do not depend on x_0 and so are conserved in time

$$\begin{aligned}I_{ab}&=\int \mathrm{d}^3x\,\psi^{\mathrm{T}}Q_{ab}\psi=\int \mathrm{d}^3x\,[(\mathrm{rot}\,\boldsymbol{H})_a(\mathrm{rot}\,\boldsymbol{H})_b\\&\qquad-(\mathrm{rot}\,\boldsymbol{E})_a(\mathrm{rot}\,\boldsymbol{E})_b+E_ap^2E_b-H_ap^2H_b]\\ \tilde{I}_{ab}&=\int \mathrm{d}^3x\,\psi^{\mathrm{T}}\tilde{Q}_{ab}\psi=\int \mathrm{d}^3x\,[E_ap^2H_b\\&\qquad+H_ap^2E_b-(\mathrm{rot}\,\boldsymbol{E})_a(\mathrm{rot}\,\boldsymbol{H})_b-(\mathrm{rot}\,\boldsymbol{H})_a(\mathrm{rot}\,\boldsymbol{E})_b].\end{aligned} \tag{5.11}$$

In contrast with the classical motion constants (energy, momentum, etc) the integral combinations (5.11) depend not only on $\boldsymbol{E}$ and $\boldsymbol{H}$ but also on the derivatives of these vectors.

So starting from the symmetry operators (5.6) found above we obtain ten new constants of motion for the electromagnetic field in vacuum given by relations (5.11). These motion constants, in contrast to the Lipkin ones (Lipkin 1964, Fradkin 1965, Kibble, 1965, Michelsson and Niederle 1984), have nothing to do with the Lorentz or conformal invariance of the Maxwell equations inasmuch as the corresponding symmetry operators (5.6) do not belong to the enveloping algebra of the algebra $\mathrm{C}(1,3)\oplus\mathrm{H}$.

Acknowledgment

We would like to thank the referee for his useful comments.

Note added in proof. In the formulation of theorem 3 we have omitted six symmetry operators of the massless Dirac equation, which have the form $Q_{\mu\nu}=-Q_{\nu\mu}=[K_\nu,A_\mu]$. So this equation has 52 linearly independent symmetry operators $Q\in\mathfrak{M}_1$. All the symmetry operators $Q\in\mathfrak{M}_1$ for the Dirac equation with $m\neq0$ belong to the enveloping algebra of algebra $P(1,3)$ inasmuch as operator B (3.3) can be represented as $D\psi=\frac{1}{4}\varepsilon_{\mu\nu\rho\sigma}J^{\mu\nu}J^{\rho\sigma}\psi$ on the set of the Dirac equation solutions.

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